高维微分学——相关分析结论

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1　知识要素

1.1　多元函数可微性的一个充分性条件

对于多元函数的可微性有如下充分性结论

定理 1.1. $\exists \boldsymbol{D}f(\boldsymbol{x}) \triangleq [\dfrac{\partial f}{\partial x^1}, \cdots, \dfrac{\partial f}{\partial x^m}](\boldsymbol{x}) \in \mathbb{R}^{1\times m}, \forall \boldsymbol{x} \in B_\lambda(x_0) \subset \mathscr{D}_x$ 且 $\dfrac{\partial f}{\partial x^i}(\boldsymbol{x}) \in \mathbb{R}$ 在 $\boldsymbol{x}_0$ 点连续 $(1 \leqslant i \leqslant m)$, 则 $f(\boldsymbol{x}) \in \mathbb{R}$ 在 $\boldsymbol{x}_0$ 点可微.

证明 按可微性定义, 需估计

$$|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - Df^\alpha(\boldsymbol{x})\boldsymbol{h}| = \left|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) - \left(\frac{\partial f}{\partial x^1}(\boldsymbol{x})h^1 + \cdots + \frac{\partial f}{\partial x^m}(\boldsymbol{x})h^m\right)\right|.$$

就此, 考虑

$$\begin{aligned}
f(\boldsymbol{x}+h) - f(\boldsymbol{x}) &= f\left(\boldsymbol{x} + h^1\boldsymbol{i}_1 + \cdots + h^m\boldsymbol{i}_m\right) - f(\boldsymbol{x}) \\
&= f(\boldsymbol{x} + h^1\boldsymbol{i}_1 + \cdots + h^m\boldsymbol{i}_m) - f(\boldsymbol{x} + h^2\boldsymbol{i}_2 + \cdots + h^m\boldsymbol{i}_m) \\
&\quad + f(\boldsymbol{x} + h^2\boldsymbol{i}_2 + \cdots + h^m\boldsymbol{i}_m) - f(\boldsymbol{x} + h^3\boldsymbol{i}_3 + \cdots + h^m\boldsymbol{i}_m) + \cdots \\
&\quad + f(\boldsymbol{x} + h^i\boldsymbol{i}_i + \cdots + h^m\boldsymbol{i}_m) - f(\boldsymbol{x} + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m) + \cdots \\
&\quad + f(\boldsymbol{x} + h^m\boldsymbol{i}_m) - f(\boldsymbol{x}),
\end{aligned}$$

由此需考虑

$$\begin{aligned}
&f\left(\boldsymbol{x} + h^i\boldsymbol{i}_i + \cdots + h^m\boldsymbol{i}_m\right) - f\left(\boldsymbol{x} + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m\right) \\
&= f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m)\Big|_{t=0}^1.
\end{aligned}$$

令

$$\phi_i(t) : [0,1] \ni t \longmapsto \phi_i(t) \triangleq f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m) \in \mathbb{R}$$

考虑

$$\begin{aligned}
&\frac{\phi_i(t+\triangle t) - \phi_i(t)}{\triangle t} \\
=&\frac{f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m + \triangle th^i\boldsymbol{i}_i) - f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m)}{\triangle t} \\
=&\frac{f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m + \triangle th^i\boldsymbol{i}_i) - f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m)}{\triangle th^i} \cdot h^i \\
&\to \frac{\partial f}{\partial x^i}(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m) \cdot h^i \quad \text{as} \, \triangle t \to 0,
\end{aligned}$$

上述最后极限存在可基于 $\exists \dfrac{\partial f}{\partial x^i}(\boldsymbol{x}) \in \mathbb{R}, \forall \boldsymbol{x} \in B_\lambda(\boldsymbol{x}_0)$.

按上所述, 如有 $\exists \dot{\phi}_i(t) \in \mathbb{R}, \forall\, t \in [0,1]$, 故由 Lagrange 中值定理有

$$\begin{aligned} &f(\boldsymbol{x} + th^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m)|_{t=0}^{1} = \phi_i(t)|_{t=0}^{1} = \dot{\phi}_i(\theta_i) \\ &= \frac{\partial f}{\partial x^i}(\boldsymbol{x} + \theta_i h^i\boldsymbol{i}_i + h^{i+1}\boldsymbol{i}_{i+1} + \cdots + h^m\boldsymbol{i}_m)h^i, \quad \theta_i \in (0,1). \end{aligned}$$

由此, 有

$$\begin{aligned} f(\boldsymbol{x}_0 + h) - f(\boldsymbol{x}_0) = &\frac{\partial f}{\partial x^1}(\boldsymbol{x}_0 + \theta_1 h^1\boldsymbol{i}_1 + \cdots + h^m\boldsymbol{i}_m)h^1 \\ &+ \frac{\partial f}{\partial x^2}(\boldsymbol{x}_0 + \theta_2 h^2\boldsymbol{i}_2 + \cdots + h^m\boldsymbol{i}_m)h^2 + \cdots \\ &+ \frac{\partial f}{\partial x^i}(\boldsymbol{x}_0 + \theta_i h^i\boldsymbol{i}_i + \cdots + h^m\boldsymbol{i}_m)h^i + \cdots \\ &+ \frac{\partial f}{\partial x^m}(\boldsymbol{x}_0 + \theta_m h^m\boldsymbol{i}_m)h^m, \quad \theta_i \in (0,1), \forall i = 1, \cdots, m. \end{aligned}$$

进一步, 有

$$\begin{aligned} &|f(\boldsymbol{x}_0 + h) - f(\boldsymbol{x}_0) - \boldsymbol{D}f(\boldsymbol{x}_0)h| \\ &\leqslant \left|\frac{\partial f}{\partial x^1}(\boldsymbol{x}_0 + \theta_1 h^1\boldsymbol{i}_1 + \cdots + h^m\boldsymbol{i}_m) - \frac{\partial f}{\partial x^1}(\boldsymbol{x}_0)\right| \cdot |h^1| \\ &+ \cdots + \left|\frac{\partial f}{\partial x^i}(\boldsymbol{x}_0 + \theta_i h^i\boldsymbol{i}_i + \cdots + h^m\boldsymbol{i}_m) - \frac{\partial f}{\partial x^i}(\boldsymbol{x}_0)\right| \cdot |h^i| \\ &+ \cdots + \left|\frac{\partial f}{\partial x^m}(\boldsymbol{x}_0 + \theta_m h^m\boldsymbol{i}_m) - \frac{\partial f}{\partial x^i}(\boldsymbol{x}_0)\right| \cdot |h^m|. \end{aligned}$$

考虑到

$$\begin{cases} \exists \lim\limits_{\boldsymbol{x} \to \boldsymbol{x}_0 \in \mathbb{R}^m} \dfrac{\partial f}{\partial x^i}(\boldsymbol{x}) = \dfrac{\partial f}{\partial x^i}(\boldsymbol{x}_0) \in \mathbb{R} \\ \exists \lim\limits_{\boldsymbol{h} \to \boldsymbol{0} \in \mathbb{R}^m} (\boldsymbol{x}_0 + \theta_i h^i\boldsymbol{i}_i + \cdots + h^m\boldsymbol{i}_m) = \boldsymbol{x}_0 \in \mathbb{R}^m \end{cases},$$

按复合向量值映照极限定理, 有

$$\exists \lim_{\boldsymbol{h} \to \boldsymbol{0} \in \mathbb{R}^m} \frac{\partial f}{\partial x^i}(\boldsymbol{x}_0 + \theta_i h^i\boldsymbol{i}_i + \cdots + h^m\boldsymbol{i}_m) = \frac{\partial f}{\partial x^i}(\boldsymbol{x}_0) \in \mathbb{R}.$$

综上, 可有

$$f(\boldsymbol{x}_0 + \boldsymbol{h}) = f(\boldsymbol{x}_0) + \boldsymbol{D}f(\boldsymbol{x}_0)h + o(\boldsymbol{h}) \in \mathbb{R}.$$

□

对于向量值映照 $\boldsymbol{f}(\boldsymbol{x}) : \mathbb{R}^m \supset \mathscr{D}_x \ni \boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x}) \in \mathbb{R}^n$ 在 $\boldsymbol{x}_0 \in int\mathscr{D}_x$ 点可微等价于因变量的所有分量 $f^\alpha(\boldsymbol{x}) : \mathbb{R}^m \supset \mathscr{D}_x \ni \boldsymbol{x} \mapsto f^\alpha(\boldsymbol{x}) \in \mathbb{R}$ 在 $\boldsymbol{x}_0$ 点可微.

1.2 多元函数偏导数可以交换次序的一个充分性条件

定理 1.2 (混合偏导数可交换次序定理).

$$\exists \quad \frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}),\ \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}) \in \mathbb{R}, \quad \forall \boldsymbol{x} \in B_\lambda(x_0)$$

且两者均在 $\boldsymbol{x}_0$ 点连续, 则有

$$\frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}_0) = \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}_0) \in \mathbb{R}.$$

证明 作函数

$$\triangle(\lambda,\mu) := f(\boldsymbol{x}_0 + \lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_j) - f(\boldsymbol{x}_0 + \lambda \boldsymbol{i}_i) - f(\boldsymbol{x}_0 + \mu \boldsymbol{i}_j) + f(\boldsymbol{x}_0).$$

引入

$$\widetilde{\phi}(t) : [0,1] \ni t \mapsto f(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_j) - f(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i) \in \mathbb{R},$$

则有

$$\triangle(\lambda,\mu) = \widetilde{\phi}(1) - \widetilde{\phi}(0) =: \widetilde{\phi}(t)|_0^1.$$

考虑到

$$\begin{aligned}
&\frac{\widetilde{\phi}(t+\triangle t) - \widetilde{\phi}(t)}{\triangle t} \\
&= \frac{f(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_j + \triangle t \lambda \boldsymbol{i}_i) - f(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_j)}{\triangle t \lambda}\lambda - \frac{f(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_i) - f(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i)}{\triangle t \lambda}\lambda \\
&\to \left[\frac{\partial f}{\partial x^i}(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_j) - \frac{\partial f}{\partial x^i}(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i)\right] \cdot \lambda =: \dot{\widetilde{\phi}}(t), \quad \text{as}\ \triangle t \mapsto 0
\end{aligned}$$

则有

$$\triangle(\lambda,\mu) = \widetilde{\phi}(t)|_0^1 = \dot{\widetilde{\phi}}(\theta) = \left[\frac{\partial f}{\partial x^i}(x_0 + \theta\lambda \boldsymbol{i}_i + \mu \boldsymbol{i}_j) - \frac{\partial f}{\partial x^i}(x_0 + \theta\lambda \boldsymbol{i}_i)\right]\lambda.$$

再引入

$$\widetilde{\psi}(t) : [0,1] \ni t \mapsto \widetilde{\psi}(t) = \frac{\partial f}{\partial x^i}(x_0 + \theta\lambda \boldsymbol{i}_i + t\mu \boldsymbol{i}_j) \in \mathbb{R},$$

可有

$$\triangle(\lambda,\mu) = \lambda \cdot \widetilde{\psi}(t)|_0^1 = \lambda\mu \frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}_0 + \theta\lambda \boldsymbol{i}_i + \widetilde{\theta}\mu \boldsymbol{i}_j) \quad \theta, \widetilde{\theta} \in (0,1).$$

综上, 有

$$\frac{\triangle(\lambda,\mu)}{\lambda\mu} = \frac{\partial^2 f}{\partial x^j \partial x^i}(x_0 + \theta\lambda \boldsymbol{i}_i + \widetilde{\theta}\mu \boldsymbol{i}_j),$$

考虑到

$$\left\{\begin{array}{l}
\exists \lim\limits_{\boldsymbol{x} \to \boldsymbol{x}_0 \in \mathbb{R}^m} \dfrac{\partial^2 f}{\partial x^j \partial x^i}(x) = \dfrac{\partial^2 f}{\partial x^j \partial x^i}(x_0) \\
\exists \lim\limits_{(\lambda,\mu) \to \boldsymbol{0} \in \mathbb{R}^2} (\boldsymbol{x}_0 + \theta\lambda \boldsymbol{i}_i + \widetilde{\theta}\mu \boldsymbol{i}_j) = \boldsymbol{x}_0 \in \mathbb{R}^m
\end{array}\right. ,$$

按复合向量值映照的极限定理, 有

$$\exists \lim_{(\lambda,\mu) \to \boldsymbol{0} \in \mathbb{R}^2} \frac{\triangle(\lambda,\mu)}{\lambda\mu} = \lim_{(\lambda,\mu) \to \boldsymbol{0} \in \mathbb{R}^2} \frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}_0 + \theta\lambda \boldsymbol{i}_i + \widetilde{\theta}\mu \boldsymbol{i}_j) = \frac{\partial^2 f}{\partial x^j \partial x^i}(\boldsymbol{x}_0).$$

类似于上述分析, 引入

$$\widehat{\phi}(t):\, [0,1] \ni t \,\mapsto\, \widehat{\phi}(t) \triangleq f(\boldsymbol{x}_0 + \lambda \boldsymbol{i}_i + t\mu \boldsymbol{i}_j) - f(\boldsymbol{x}_0 + t\mu \boldsymbol{i}_j),$$

可有

$$\triangle(\lambda,\mu) = \widehat{\phi}(t)|_0^1 = \dot{\widehat{\phi}}(\theta_1) = \left[\frac{\partial f}{\partial x^j}(\boldsymbol{x}_0 + \lambda \boldsymbol{i}_i + \theta_1 \mu \boldsymbol{i}_j) - \frac{\partial f}{\partial x^j}(\boldsymbol{x}_0 + \theta_1 \mu \boldsymbol{i}_j)\right]\mu.$$

再引入

$$\widehat{\psi}(t):\, [0,1] \ni t \,\mapsto\, \widehat{\psi}(t) \triangleq \frac{\partial f}{\partial x^j}(\boldsymbol{x}_0 + t\lambda \boldsymbol{i}_i + \theta_1 \mu \boldsymbol{i}_j) \in \mathbb{R},$$

可有

$$\triangle(\lambda,\mu) = \mu\widehat{\psi}(t)_0^1 = \mu\dot{\widehat{\psi}}(\theta_2) = \lambda\mu\frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}_0 + \theta_2 \lambda \boldsymbol{i}_i + \theta_1 \mu \boldsymbol{i}_j).$$

综上, 有

$$\frac{\triangle(\lambda,\mu)}{\lambda\mu} = \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}_0 + \theta_2 \lambda \boldsymbol{i}_i + \theta_1 \mu \boldsymbol{i}_j),$$

考虑到

$$\begin{cases} \exists \lim\limits_{\boldsymbol{x}\to\boldsymbol{x}_0\in\mathbb{R}^m} \dfrac{\partial^2 f}{\partial x^i \partial x^j}(x) = \dfrac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}_0) \\ \exists \lim\limits_{(\lambda,\mu)\to\boldsymbol{0}\in\mathbb{R}^2} (\boldsymbol{x}_0 + \theta_2 \lambda \boldsymbol{i}_i + \theta_1 \mu \boldsymbol{i}_j) = \boldsymbol{x}_0 \in \mathbb{R}^m \end{cases},$$

按复合向量值映照的极限定理, 有

$$\exists \lim_{(\lambda,\mu)\to\boldsymbol{0}\in\mathbb{R}^2} \frac{\triangle(\lambda,\mu)}{\lambda\mu} = \lim_{(\lambda,\mu)\to\boldsymbol{0}\in\mathbb{R}^2} \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}_0 + \theta_2 \lambda \boldsymbol{i}_i + \theta_1 \mu \boldsymbol{i}_j) = \frac{\partial^2 f}{\partial x^i \partial x^j}(\boldsymbol{x}_0).$$

□

2 应用事例

3 建立路径

主要的分析思想表现为将多元函数 $f(\boldsymbol{x}) : \mathbb{R}^m \supset \mathscr{D}_x \ni \boldsymbol{x} \mapsto f(\boldsymbol{x}) \in \mathbb{R}$ 限制在直线段 $[\boldsymbol{a}, \boldsymbol{b}] := \{\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}) \,|\, t \subset [0,1]\}$ 上, 就此可以引入一元函数

$$\phi(t):\, [0,1] \ni t \,\mapsto\, f(\boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})) \in \mathbb{R}.$$

如果 $f(\boldsymbol{x})$ 在 $[\boldsymbol{a}, \boldsymbol{b}]$ 上连续, 且在直线段内部存在关于 $\boldsymbol{b} - \boldsymbol{a}$ 的方向导数, 则对 $\phi(t)$ 在 $[0,1]$ 上可利用 Lagrange 中值定理

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = \phi(1) - \phi(0) = \dot{\phi}(\theta) = \frac{\partial f}{\partial \boldsymbol{e}}(\boldsymbol{a} + \theta \cdot (\boldsymbol{b} - \boldsymbol{a}))|\boldsymbol{b} - \boldsymbol{a}|_{\mathbb{R}^m}, \quad \theta \subset (0,1),$$

式中 $\boldsymbol{e} := \dfrac{\boldsymbol{b} - \boldsymbol{a}}{|\boldsymbol{b} - \boldsymbol{a}|_{\mathbb{R}^m}}$. 上述分析可隶属**数学通识**.