复旦大学力学与工程科学系

2011～2012学年第二学期期末考试

□A卷　　　□B卷

课程名称：连续介质力学基础　　　课程代码：MECH130105

开课院系：力学与工程科学系　　　考试形式：开卷/闭卷/课程论文

姓名：______________　学号：______________　专业：______________

题 号	1/(1)	1/(2)	2/(1)	2/(2)	2/(3)	3/(1)	3/(2)	3/(3)	3/(4)	3/(5)
得 分										
题 号	4/(1)	4/(2)	5/(1)	5/(2)	5/(3)	5/(4)	5/(5)	5/(6)	5/(7)	总 分
得 分										

Problem 1 (To study the Stokes theorem) *The general relations of the curve integrals and surface integrals could be generally term as Stokes theorem. It is seemed that all kinds of the relations could be concluded as two forms as shown below*

$$\oint_C \tau \circ - \overset{\Sigma}{\mathbf{\Phi}}\, dl = \int_\Sigma \left[(\mathbf{n}\times\nabla)\circ - \mathbf{\Phi}\right]_\Sigma d\sigma$$

$$\oint_C (\tau\times\mathbf{n}) \circ - \overset{\Sigma}{\mathbf{\Phi}}\, dl$$

$$= \int_\Sigma \left[\nabla\circ - \mathbf{\Phi} - (\nabla\cdot\mathbf{n})\,(\mathbf{n}\circ - \mathbf{\Phi}) - \mathbf{n}\circ - (\mathbf{n}\cdot(\nabla\otimes\mathbf{\Phi}))\right]_\Sigma d\sigma$$

where $\nabla := \mathbf{i}\,\frac{\partial}{\partial X^\alpha}$ *is the general three dimensional gradient operator. We term these relations as the generalized Stokes formulas of the first kind and the second kind respectively.*

1. *To prove the generalized Stokes formula of the second kind.*

2. *Furthermore, it could be proved that*

$$\oint_C (\tau\times\mathbf{n}) \circ - \overset{\Sigma}{\mathbf{\Phi}}\, dl = \int_\Sigma \left(\overset{\Sigma}{\nabla}\circ - \overset{\Sigma}{\mathbf{\Phi}}\right) + H\left(\mathbf{n}\circ - \overset{\Sigma}{\mathbf{\Phi}}\right) d\sigma$$

where $\overset{\Sigma}{\nabla} := \overset{\Sigma}{\mathbf{g}}{}^s \frac{\partial}{\partial x^s_\Sigma}$ *is always termed as the gradient operator on the surface, and* H *denotes the mean curvature. It indicates that the surface integrand is independent on the three dimensionalization.*

Problem 2 (Some Properties of a tensor represented in multi-points form) *Generally, any tensor could be represented in multi-points form that is the composition of the corresponding simple tensor are coming from different bases.*

1. *The so-called transfer tensor is defined as*

$$\mathbf{I} = g^i_A \, \mathbf{g}_i \otimes \mathbf{G}^A, \quad g^i_A := (\mathbf{g}^i, \mathbf{G}_A)$$

where $\{\mathbf{g}_i\}_{i=1}^m$ *and* $\{\mathbf{G}_A\}_{A=1}^m$ *are arbitrary two bases in* $\mathbb{R}^m$*. To prove*

$$g^i_A \, \mathbf{g}_i \otimes \mathbf{G}^A = g^{ij} \, \mathbf{g}_i \otimes \mathbf{g}_j = G_{AB} \, \mathbf{G}^A \otimes \mathbf{G}^B$$

2. *There exists the represents of a tenor in triple-points form, say*

$$\mathbf{\Phi} = \Phi^{i\cdot\xi}_{\cdot A}(x,y,z)\overset{x}{\mathbf{g}}_i(x) \otimes \overset{y}{\mathbf{g}}{}^A(y) \otimes \overset{z}{\mathbf{g}}_\xi(z) \in \mathscr{T}^3(\mathbb{R}^m)$$

where $\{\overset{x}{\mathbf{g}}_i\}_{i=1}^m(x)$, $\{\overset{y}{\mathbf{g}}_A\}_{A=1}^m(y)$ *and* $\{\overset{z}{\mathbf{g}}_\xi\}_{\xi=1}^m(z)$ *could be taken as three local bases in* $\mathbb{R}^m$ *with respect to three curvilinear coordinates/diffeomorphisms denoted by* $X(x)$, $X(y)$ *and* $X(z)$ *correspondingly. The relations between the curvilinear coordinates are diffeomorphisms mutually thanks to the transfer property between diffeomorphisms. Therefore, one could calculate*

$$\frac{\partial \mathbf{\Phi}}{\partial y^L}(y) =: \overset{y}{\square}_L \Phi^{i\cdot\xi}_{\cdot A} \, \overset{x}{\mathbf{g}}_i(x) \otimes \overset{y}{\mathbf{g}}{}^A(y) \otimes \overset{z}{\mathbf{g}}_\xi(z) \in \mathscr{T}^3(\mathbb{R}^m)$$

in the relation

$$\overset{y}{\square}_L \Phi^{i\cdot\xi}_{\cdot A} := \overset{y}{\nabla}_L \Phi^{i\cdot\xi}_{\cdot A} + \frac{\partial x^l}{\partial y^L}(y) \, \overset{x}{\nabla}_l \Phi^{i\cdot\xi}_{\cdot A} + \frac{\partial z^\theta}{\partial y^L}(y) \, \overset{z}{\nabla}_\theta \Phi^{i\cdot\xi}_{\cdot A}$$

where $\overset{x}{\nabla}_l$, $\overset{y}{\nabla}_L$ *and* $\overset{z}{\nabla}_\theta$ *are general covariant derivatives with respect to the curvilinear coordinates* $X(x)$, $X(y)$ *and* $X(z)$ *respectively. To prove the above assertion.*

3. *Furthermore, one could introduced the corresponding anholonomic bases denoted by* $\{\overset{x}{\mathbf{g}}_{(i)}\}_{i=1}^m(x)$, $\{\overset{y}{\mathbf{g}}_{(A)}\}_{A=1}^m(y)$ *and* $\{\overset{z}{\mathbf{g}}_{(\xi)}\}_{\xi=1}^m(z)$ *respectively. Then, one has*

$$\frac{\partial \mathbf{\Phi}}{\partial y^L}(y) =: \overset{y}{\square}_L \Phi^{i\cdot\xi}_{\cdot A} \, \overset{x}{\mathbf{g}}_i(x) \otimes \overset{y}{\mathbf{g}}{}^A(y) \otimes \overset{z}{\mathbf{g}}_\xi(z) = \overset{y}{\square}_{(L)} \Phi^{(i)\cdot(\xi)}_{\cdot(A)} \, \overset{x}{\mathbf{g}}_{(i)}(x) \otimes \overset{y}{\mathbf{g}}{}^{(A)}(y) \otimes \overset{z}{\mathbf{g}}_{(\xi)}(z)$$

in the relation

$$\overset{y}{\square}_{(L)} \Phi^{(i)\cdot(\xi)}_{\cdot(A)} := \overset{y}{\nabla}_{(L)} \Phi^{(i)\cdot(\xi)}_{\cdot(A)} + \frac{\partial x^{(l)}}{\partial y^{(L)}}(y) \, \overset{x}{\nabla}_{(l)} \Phi^{(i)\cdot(\xi)}_{\cdot(A)} + \frac{\partial z^{(\theta)}}{\partial y^{(L)}}(y) \, \overset{z}{\nabla}_\theta \Phi^{(i)\cdot(\xi)}_{\cdot(A)}$$

where

$$\frac{\partial x^{(l)}}{\partial y^{(L)}}(y) := C^{(l)}_k \, C^D_{(L)} \, \frac{\partial x^k}{\partial y^D}(y), \quad \frac{\partial z^{(\theta)}}{\partial y^{(L)}}(y) := C^{(\theta)}_\gamma \, C^D_{(L)} \, \frac{\partial z^\gamma}{\partial y^D}(y)$$

and $\overset{y}{\nabla}_{(L)} \Phi^{(i)\cdot(\xi)}_{\cdot(A)}$ *is the covariant derivative with respect to the anholonomic basis* $\{\overset{y}{\mathbf{g}}_{(A)}\}_{A=1}^m(y)$.

Problem 3 (To study the deformation gradient tensor) *The relationship between the vectors connecting the same point* $\mathbf{a}$ *and* $\mathbf{b}$ *in the current physical configuration and initial physical configuration, denoted by* $\mathbf{r}_{ab}|_{\overset{t}{V}}$ *and* $\mathbf{r}_{ab}|_{\overset{\circ}{V}}$ *respectively, could be represented as follows:*

$$\mathbf{r}_{ab}|_{\overset{t}{V}} = \mathbf{F} \cdot \mathbf{r}_{ab}|_{\overset{\circ}{V}}, \quad \textit{where}\, \mathbf{F} \triangleq \left[\frac{\partial x^i}{\partial \xi^A}(\xi,t)\mathbf{g}_i(x,t) \otimes G^A(\xi)\right]$$

where $\mathbf{F}$ *is termed as the deformation gradient tensor. Its*

1. *To deduce the representation of the deformation gradient tensor.*

2. *To prove that its determinant* $\det \mathbf{F} \equiv |\mathbf{F}|$ *is*

$$\det \mathbf{F} = \frac{\sqrt{g}}{\sqrt{G}} \cdot \det \left[\frac{\partial x^i}{\partial \xi^A}\right](\xi,t) := |\mathbf{F}|$$

3. *To prove that its material derivative is*

$$\dot{\mathbf{F}} = (\mathbf{V} \otimes \square) \cdot \mathbf{F}$$

4. *To prove that the material derivative of its determinant is*

$$\dot{\overline{|\mathbf{F}|}} = \theta|\mathbf{F}|, \quad \textit{where}\, \theta \triangleq \square \cdot \mathbf{V}$$

5. *To prove the transport theorem of the continuous media whose geometrical status is volume, that is*

$$\frac{d}{dt}\int_{\overset{t}{\mathscr{V}}} \Phi(x,t)d\tau = \int_{\overset{t}{\mathscr{V}}} (\dot{\Phi} + \theta\Phi)d\tau = \int_{\overset{t}{\mathscr{V}}} \frac{\partial \Phi}{\partial t}(x,t)d\tau + \oint_{\partial\overset{t}{\mathscr{V}}} \Phi(v \cdot n)d\sigma$$

Problem 4 (Governing equations in Lagrangian arguments) *The governing equations of continuum media represented in Lagrangian arguments could be listed as follows*

mass conservation	$\rho(\xi,t)\,|\mathbf{F}|(\xi,t) = \overset{\circ}{\rho}(\xi)$
momentum conservation	$\overset{\circ}{\rho}(\xi)\,\mathbf{a}(\xi,t) = \overset{\circ}{\rho}(\xi)\,\mathbf{f}_m(\xi,t) + \begin{cases} [\mathbf{t} \cdot (|\mathbf{F}|\mathbf{F}^{-*})] \cdot \overset{\circ}{\square} =: \tau \cdot \overset{\circ}{\square} \\ (\mathbf{F} \cdot \mathbf{T}) \cdot \overset{\circ}{\square}, \quad \mathbf{T} = \mathbf{F}^{-1} \cdot \tau \end{cases}$
constitution relation	$\mathbf{T} = -p\,\mathbf{C}_L^{-1} + 2\left[\frac{\partial \Sigma}{\partial I_1}\mathbf{I} + \frac{\partial \Sigma}{\partial I_2}(I_1\mathbf{I} - \mathbf{C}_L)\right], \quad \mathbf{C}_L := \mathbf{F}^* \cdot \mathbf{F}$

where the constitution relation is corresponding to the incompressible flow.

1. *To prove the equation of mass conservation.*

2. *To prove the equation of momentum conservation.*

Problem 5 (To study the dynamics of a rotating annulus with finite deformation) *The curvilinear coordinates with respect to the initial and current configurations are the general cylindrical coordinates. In the case of the rotation with a fixed angular velocity, the description of the motion is assumed that*

$$\begin{bmatrix} r \\ \theta \\ z \end{bmatrix} (R, \Theta, Z, t) \triangleq \begin{bmatrix} r(R) \\ \Theta + \omega\, t \\ \lambda\, Z \end{bmatrix}$$

The initial inner and outer radii is denoted as a and b respectively.

1. *To analysis the equation of mass conservation under the assumption that the deformation is incompressible.*
2. *To deduce the component form of the constitution relation.*
3. *To deduce the component form of the momentum equation.*
4. *To solve the pressure distribution.*
5. *To solve the distributions of the stress.*
6. *In the general case of the rotation, the description of the motion is assumed that*

$$\begin{bmatrix} r \\ \theta \\ z \end{bmatrix} (R, \Theta, Z, t) \triangleq \begin{bmatrix} r(R, t) \\ \Theta + \int_0^t \dot{\omega}(\tau)\, d\tau \\ z(Z, t) \end{bmatrix}$$

   To carry out the related analysis on mass conservation.

7. *To carry out the related analysis on momentum conservation with constitution relation.*

Note: To give the deduction and calculation in detail. And as the score is considered, the reflection of the correct methodologies is oriented.