曲面上标架运动方程

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1　知识要素

1.1　曲面上标架运动方程

我们约定, 小写英文字母指标的求和范围为 1 至 m, 小写希腊字母指标的求和范围为 1 至 $m+1$.

研究协变基向量沿坐标线的变化率，有

$$\begin{aligned}\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) &\triangleq \lim_{\lambda\to 0\in\mathbb{R}} \frac{\boldsymbol{g}_i(\boldsymbol{x}_\Sigma+\lambda \boldsymbol{i}_j)-\boldsymbol{g}_i(\boldsymbol{x}_\Sigma)}{\lambda} \\ &= \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}_k\right)_{\mathbb{R}^{m+1}} \boldsymbol{g}^k + \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{n}\right)_{\mathbb{R}^{m+1}} \boldsymbol{n} \\ &= \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}^k\right)_{\mathbb{R}^{m+1}} \boldsymbol{g}_k + \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{n}\right)_{\mathbb{R}^{m+1}} \boldsymbol{n}.\end{aligned}$$

此处引入曲面上的 Christoffel 符号, 定义为

$$\Gamma_{ij,k} = \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}_k\right)_{\mathbb{R}^{m+1}}, \quad \Gamma_{ij}^k = \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}^k\right)_{\mathbb{R}^{m+1}}.$$

$\{\Gamma_{ij,k}\}_{i,j,k=1}^m$ 称为**曲面的第一类 Christoffel 符号**, $\{\Gamma_{ij}^k\}_{i,j,k=1}^m$ 为**曲面的第二类 Christoffel 符号**.

因此对于切向量, 有

$$\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) = \Gamma_{ij,k}\boldsymbol{g}^k + b_{ij}\boldsymbol{n} = \Gamma_{ij}^k \boldsymbol{g}_k + b_{ij}\boldsymbol{n}.$$

同样地, 研究法向量沿坐标线的变化率, 有

$$\begin{aligned}\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) &\triangleq \lim_{\lambda\to 0\in\mathbb{R}} \frac{\boldsymbol{n}(\boldsymbol{x}_\Sigma+\lambda \boldsymbol{i}_j)-\boldsymbol{n}(\boldsymbol{x}_\Sigma)}{\lambda} \\ &= \left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}_k\right)_{\mathbb{R}^{m+1}} \boldsymbol{g}^k + \left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{n}\right)_{\mathbb{R}^{m+1}} \boldsymbol{n} \\ &= \left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}^k\right)_{\mathbb{R}^{m+1}} \boldsymbol{g}_k + \left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{n}\right)_{\mathbb{R}^{m+1}} \boldsymbol{n},\end{aligned}$$

式中

$$\left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{n}\right)_{\mathbb{R}^{m+1}} = \frac{1}{2}\frac{\partial}{\partial x_\Sigma^j}(\boldsymbol{n}, \boldsymbol{n})_{\mathbb{R}^{m+1}} = 0;$$

$$\left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}_k\right)_{\mathbb{R}^{m+1}} = \frac{\partial}{\partial x_\Sigma^j}(\boldsymbol{n}, \boldsymbol{g}_k)_{\mathbb{R}^{m+1}} - \left(\boldsymbol{n}, \frac{\partial \boldsymbol{g}_k}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma)\right)_{\mathbb{R}^{m+1}} = -b_{jk};$$

$$\left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), \boldsymbol{g}^k\right)_{\mathbb{R}^{m+1}} = \left(\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma), g^{kt}\boldsymbol{g}_t\right)_{\mathbb{R}^{m+1}} = -g^{kt}b_{jt} = -b_j^k.$$

因此对于法向量有

$$\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) = -b_{jk}\boldsymbol{g}^k = -b_j^k\boldsymbol{g}_k.$$

综上, 协变基标架的标架运动方程为

$$\begin{cases} \dfrac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) = \Gamma_{ij}^k\boldsymbol{g}_k + b_{ij}\boldsymbol{n} = \Gamma_{ij,k}\boldsymbol{g}^k + b_{ij}\boldsymbol{n}; \\ \dfrac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) = -b_{jk}\boldsymbol{g}^k = -b_j^k\boldsymbol{g}_k. \end{cases}$$

同理可得, 逆变基标架的标架运动方程为

$$\begin{cases} \dfrac{\partial \boldsymbol{g}^i}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) = -\Gamma_{jk}^i\boldsymbol{g}^k + b_j^i\boldsymbol{n}; \\ \dfrac{\partial \boldsymbol{n}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma) = -b_{jk}\boldsymbol{g}^k = -b_j^k\boldsymbol{g}_k. \end{cases}$$

性质 1.1 (曲面上 Christoffel 符号的基本性质)**.** 与 Euclid 空间中的 Christoffel 符号类似, 曲面上的 Christoffel 符号也具有如下性质.

1. 第一类 Christoffel 符号同度量张量之间的关系:

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_\Sigma^j} + \frac{\partial g_{jk}}{\partial x_\Sigma^i} - \frac{\partial g_{ij}}{\partial x_\Sigma^k}\right)(\boldsymbol{x}_\Sigma);$$

2. 第二类 Christoffel 符号同度量张量之间的关系:

$$\Gamma_{ij}^i \triangleq g^{ik}\Gamma_{ij,k} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma);$$

3. 对于高维曲面有

$$g^{kl}\Gamma_{kl}^i = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\Sigma^k}(\sqrt{g}g^{ik}).$$

证明 通过直接计算, 可证明曲面上的 Christoffel 符号的基本性质.

1. 此关系的证明完全类似于体积上对应结论的处理. 主要基于结构

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x_\Sigma^l} &= \frac{\partial}{\partial x_\Sigma^l}\left(\boldsymbol{g}_i, \boldsymbol{g}_j\right)_{\mathbb{R}^{m+1}}(\boldsymbol{x}_\Sigma) \\ &= \left(\frac{\partial \boldsymbol{g}_i}{\partial x_\Sigma^l}, \boldsymbol{g}_j\right)_{\mathbb{R}^{m+1}}(\boldsymbol{x}_\Sigma) + \left(\boldsymbol{g}_i, \frac{\partial \boldsymbol{g}_j}{\partial x_\Sigma^l}\right)_{\mathbb{R}^{m+1}}(\boldsymbol{x}_\Sigma) = \Gamma_{li,j} + \Gamma_{lj,i} \end{aligned}$$

然后, 利用指标轮换即得证.

2. 利用上述结论, 以及曲面度量行列式的引理, 有

$$
\begin{aligned}
\Gamma_{ij}^i &= g^{ik}\Gamma_{ij,k} \\
&= g^{ik}\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_\Sigma^j}+\frac{\partial g_{jk}}{\partial x_\Sigma^i}-\frac{\partial g_{ij}}{\partial x_\Sigma^k}\right)(\boldsymbol{x}_\Sigma) \\
&= \frac{1}{2}g^{ik}\frac{\partial g_{ik}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma)=\frac{1}{2}\frac{1}{g}\frac{\partial g}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma)=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_\Sigma^j}(\boldsymbol{x}_\Sigma).
\end{aligned}
$$

3. 考虑

$$
\begin{aligned}
g^{kl}\Gamma_{kl}^i = g^{kl}g^{is}\Gamma_{kl,s} &= g^{kl}g^{is}\frac{1}{2}\left(\frac{\partial g_{ks}}{\partial x_\Sigma^l}+\frac{\partial g_{ls}}{\partial x_\Sigma^k}-\frac{\partial g_{kl}}{\partial x_\Sigma^s}\right) \\
&= \frac{1}{2}g^{kl}g^{is}\left(\frac{\partial g_{ks}}{\partial x_\Sigma^l}+\frac{\partial g_{ls}}{\partial x_\Sigma^k}\right)-\frac{1}{2}g^{is}g^{kl}\frac{\partial g_{kl}}{\partial x_\Sigma^s} \\
&= g^{kl}g^{is}\frac{\partial g_{ks}}{\partial x_\Sigma^l}-\frac{1}{2}g^{is}g^{kl}\frac{\partial g_{kl}}{\partial x_\Sigma^s}.
\end{aligned}
$$

对上式右端第一项, 有

$$
g^{kl}g^{is}\frac{\partial g_{ks}}{\partial x_\Sigma^l}=g^{kl}\left[\frac{\partial}{\partial x_\Sigma^l}(g^{is}g_{ks})-\frac{\partial g^{is}}{\partial x_\Sigma^l}g_{ks}\right]=-\delta_s^l\frac{\partial g^{is}}{\partial x_\Sigma^l}=-\frac{\partial g^{is}}{\partial x_\Sigma^s};
$$

对右端第二项, 有

$$
-\frac{1}{2}g^{is}g^{kl}\frac{\partial g_{kl}}{\partial x_\Sigma^s}=-\frac{1}{2}g^{is}\frac{1}{g}\frac{\partial g}{\partial x_\Sigma^s}=-g^{is}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_\Sigma^s}.
$$

所以

$$
g^{kl}\Gamma_{kl}^i=-\frac{\partial g^{is}}{\partial x_\Sigma^s}-g^{is}\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x_\Sigma^s}=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\Sigma^s}(\sqrt{g}g^{is}).
$$

□

1.2 曲面局部参数化

由曲面的一般参数表示

$$
\mathbb{R}^m\supset D_x\ni\boldsymbol{x}_\Sigma=\begin{pmatrix}x_\Sigma^1\\ \vdots\\ x_\Sigma^m\end{pmatrix}\mapsto\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma)=\begin{pmatrix}X^1\\ \vdots\\ X^{m+1}\end{pmatrix}\in\mathbb{R}^{m+1},
$$

可有 $\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma)$ 各个分量的无限小增量公式:

$$
\begin{aligned}
X^\alpha(\mathring{\boldsymbol{x}}_\Sigma+\Delta\boldsymbol{x}_\Sigma) &= X^\alpha(\mathring{\boldsymbol{x}}_\Sigma)+DX^\alpha(\mathring{\boldsymbol{x}}_\Sigma)\Delta\boldsymbol{x}_\Sigma+\frac{1}{2}(\Delta\boldsymbol{x}_\Sigma)^{\mathrm{T}}H_{X^\alpha}(\mathring{\boldsymbol{x}}_\Sigma)\Delta\boldsymbol{x}_\Sigma+o^\alpha(|\Delta\boldsymbol{x}_\Sigma|_{\mathbb{R}^m}^2) \\
&= X^\alpha(\mathring{\boldsymbol{x}}_\Sigma)+\Delta x_\Sigma^i\frac{\partial X^\alpha}{\partial x_\Sigma^i}(\mathring{\boldsymbol{x}}_\Sigma)+\frac{1}{2}\frac{\partial^2X^\alpha}{\partial x_\Sigma^i\partial x_\Sigma^j}(\mathring{\boldsymbol{x}}_\Sigma)\Delta x_\Sigma^i\Delta x_\Sigma^j+o^\alpha(|\Delta\boldsymbol{x}_\Sigma|_{\mathbb{R}^m}^2),
\end{aligned}
$$

故有

$$\begin{aligned}
\boldsymbol{\Sigma}(\mathring{\boldsymbol{x}}_\Sigma+\Delta x_\Sigma) &= \boldsymbol{\Sigma}(\mathring{\boldsymbol{x}}_\Sigma)+\Delta x_\Sigma^i\frac{\partial X^\alpha}{\partial x_\Sigma^i}(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{i}_\alpha+\frac{1}{2}\Delta x_\Sigma^i\Delta x_\Sigma^j\frac{\partial^2 X^\alpha}{\partial x_\Sigma^i\partial x_\Sigma^j}(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{i}_\alpha \\
&\quad + o^\alpha(|\Delta\boldsymbol{x}_\Sigma|^2_{\mathbb{R}^m})\boldsymbol{i}_\alpha \\
&= \boldsymbol{\Sigma}(\mathring{\boldsymbol{x}}_\Sigma)+\Delta x_\Sigma^i\boldsymbol{g}_i(\mathring{\boldsymbol{x}}_\Sigma)+\frac{1}{2}\Delta x_\Sigma^i\Delta x_\Sigma^j\frac{\partial\boldsymbol{g}_j}{\partial x_\Sigma^i}(\mathring{\boldsymbol{x}}_\Sigma)+\boldsymbol{o}(|\Delta\boldsymbol{x}_\Sigma|^2_{\mathbb{R}^m}) \\
&= \boldsymbol{\Sigma}(\mathring{\boldsymbol{x}}_\Sigma)+\Delta x_\Sigma^i\boldsymbol{g}_i(\mathring{\boldsymbol{x}}_\Sigma)+\frac{1}{2}\Delta x_\Sigma^i\Delta x_\Sigma^j\left(\mathit{\Gamma}_{ij}^k(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{g}_k(\mathring{\boldsymbol{x}}_\Sigma)+b_{ij}(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{n}(\mathring{\boldsymbol{x}}_\Sigma)\right) \\
&\quad + \boldsymbol{o}(|\Delta\boldsymbol{x}_\Sigma|^2_{\mathbb{R}^m}) \\
&= \boldsymbol{\Sigma}(\mathring{\boldsymbol{x}}_\Sigma)+\left(\Delta x_\Sigma^k+\frac{1}{2}\mathit{\Gamma}_{ij}^k(\mathring{\boldsymbol{x}}_\Sigma)\Delta x_\Sigma^i\Delta x_\Sigma^j\right)\boldsymbol{g}_k(\mathring{\boldsymbol{x}}_\Sigma) \\
&\quad + \frac{1}{2}b_{ij}(\mathring{\boldsymbol{x}}_\Sigma)\Delta x_\Sigma^i\Delta x_\Sigma^j\boldsymbol{n}(\mathring{\boldsymbol{x}}_\Sigma)+\boldsymbol{o}(|\Delta\boldsymbol{x}_\Sigma|^2_{\mathbb{R}^m}).
\end{aligned}$$

由于 $\{\boldsymbol{g}_k(\mathring{\boldsymbol{x}}_\Sigma)\}_{k=1}^m$ 非正交, 故由上述形式不便获得曲面的局部形态. 考虑到 $\exists\,\boldsymbol{S}(\mathring{\boldsymbol{x}}_\Sigma)\in\mathbb{R}^{m\times m}$ 非奇异, 满足

$$\boldsymbol{S}^{\mathrm{T}}\left(g_{ij}\right)(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{S}=\boldsymbol{I}_m,\ \boldsymbol{S}^{\mathrm{T}}\left(b_{ij}\right)(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{S}=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_m\end{pmatrix},$$

定义

$$\begin{pmatrix}\hat{\boldsymbol{g}}_1 & \cdots & \hat{\boldsymbol{g}}_m\end{pmatrix}\triangleq\begin{pmatrix}\boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m\end{pmatrix}\boldsymbol{S},$$

则 $\{\hat{\boldsymbol{g}}\}_{i=1}^m$ 为切空间 $T_x\Sigma$ 的单位正交基. 故引入另一参数坐标 $\boldsymbol{y}_\Sigma=\boldsymbol{S}^{-1}\boldsymbol{x}_\Sigma$. 由于参数间的变化为线性变换, $\boldsymbol{y}_\Sigma$ 同 $\boldsymbol{x}_\Sigma$ 有全局意义的微分同胚存在, 因此可有

$$\hat{\boldsymbol{\Sigma}}(\boldsymbol{y}_\Sigma)\triangleq\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma(\boldsymbol{y}_\Sigma))=\boldsymbol{\Sigma}(\boldsymbol{S}\boldsymbol{y}_\Sigma).$$

于是

$$\begin{pmatrix}\hat{\boldsymbol{g}}_1 & \cdots & \hat{\boldsymbol{g}}_m\end{pmatrix}(\boldsymbol{y}_\Sigma)\triangleq D\hat{\boldsymbol{\Sigma}}(\boldsymbol{y}_\Sigma)=D\boldsymbol{\Sigma}(\boldsymbol{x}_\Sigma)\boldsymbol{S}=\begin{pmatrix}\boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m\end{pmatrix}(\boldsymbol{x}_\Sigma)\boldsymbol{S},$$

其中 $\{\hat{\boldsymbol{g}}_i\}_{i=1}^m(\mathring{\boldsymbol{y}}_\Sigma)$ 为单位正交基, 对应 $\left(\hat{g}_{ij}\right)(\mathring{\boldsymbol{y}}_\Sigma)=\boldsymbol{I}_m\in\mathbb{R}^{m\times m}$.

另有

$$\hat{b}_{ij}(\mathring{\boldsymbol{y}}_\Sigma)\triangleq\left(\frac{\partial\hat{\boldsymbol{g}}_j}{\partial y_\Sigma^i}(\mathring{\boldsymbol{y}}_\Sigma),\hat{\boldsymbol{n}}(\mathring{\boldsymbol{y}}_\Sigma)\right)_{\mathbb{R}^{m+1}}=-\left(\hat{\boldsymbol{g}}_j(\mathring{\boldsymbol{y}}_\Sigma),\frac{\partial\hat{\boldsymbol{n}}}{\partial y_\Sigma^i}(\mathring{\boldsymbol{y}}_\Sigma)\right)_{\mathbb{R}^{m+1}},$$

$$\left(\hat{b}_{ij}\right)(\mathring{\boldsymbol{y}}_\Sigma)=-\begin{pmatrix}\hat{\boldsymbol{g}}_1^{\mathrm{T}}\\ \vdots\\ \hat{\boldsymbol{g}}_m^{\mathrm{T}}\end{pmatrix}(\mathring{\boldsymbol{y}}_\Sigma)\begin{pmatrix}\dfrac{\partial\hat{\boldsymbol{n}}}{\partial y_\Sigma^1} & \cdots & \dfrac{\partial\hat{\boldsymbol{n}}}{\partial y_\Sigma^m}\end{pmatrix}(\mathring{\boldsymbol{y}}_\Sigma)=-(D\hat{\boldsymbol{\Sigma}})^{\mathrm{T}}(\mathring{\boldsymbol{y}}_\Sigma)D\hat{\boldsymbol{n}}(\mathring{\boldsymbol{y}}_\Sigma),$$

式中 $\hat{\boldsymbol{n}}(\boldsymbol{y}_\Sigma)=\boldsymbol{n}(\boldsymbol{x}_\Sigma(\boldsymbol{y}_\Sigma))$. 考虑到

$$D\hat{\boldsymbol{n}}(\boldsymbol{y}_\Sigma)=D\boldsymbol{n}(\boldsymbol{x}_\Sigma)D\boldsymbol{x}_\Sigma(\boldsymbol{y}_\Sigma)=D\boldsymbol{n}(\boldsymbol{x}_\Sigma)\boldsymbol{S},$$

故有

$$\left(\hat{b}_{ij}\right)(\mathring{\boldsymbol{y}}_\Sigma) = -\boldsymbol{S}^{\mathrm{T}}(D\boldsymbol{\Sigma})^{\mathrm{T}}(\mathring{\boldsymbol{x}}_\Sigma)D\boldsymbol{n}(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}}\left(b_{ij}\right)(\mathring{\boldsymbol{x}}_\Sigma)\boldsymbol{S} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}.$$

综上, 可有

$$\begin{aligned} \hat{\boldsymbol{\Sigma}}(\mathring{\boldsymbol{y}}_\Sigma + \Delta\boldsymbol{y}_\Sigma) &= \hat{\boldsymbol{\Sigma}}(\mathring{\boldsymbol{y}}_\Sigma) + \left(\Delta y_\Sigma^k + \frac{1}{2}\hat{\Gamma}_{ij}^k(\mathring{\boldsymbol{y}}_\Sigma)\Delta y_\Sigma^i \Delta y_\Sigma^j\right)\hat{\boldsymbol{g}}_k(\mathring{\boldsymbol{y}}_\Sigma) \\ &\quad + \frac{1}{2}(\Delta\boldsymbol{y}_\Sigma)^{\mathrm{T}}\left(\hat{b}_{ij}\right)(\mathring{\boldsymbol{y}}_\Sigma)\Delta\boldsymbol{y}_\Sigma\hat{\boldsymbol{n}}(\mathring{\boldsymbol{y}}_\Sigma) + \boldsymbol{o}(|\Delta\boldsymbol{y}_\Sigma|_{\mathbb{R}^m}^2) \\ &= \hat{\boldsymbol{\Sigma}}(\mathring{\boldsymbol{y}}_\Sigma) + \left[\Delta y_\Sigma^k + o^k(|\Delta\boldsymbol{y}_\Sigma|_{\mathbb{R}^{m+1}})\right]\hat{\boldsymbol{g}}_k(\mathring{\boldsymbol{y}}_\Sigma) + \frac{1}{2}\lambda_k(\Delta y_\Sigma^k)^2\hat{\boldsymbol{n}}(\mathring{\boldsymbol{y}}_\Sigma) \\ &\quad + \boldsymbol{o}(|\Delta\boldsymbol{y}_\Sigma|_{\mathbb{R}^m}^2). \end{aligned}$$

现以 $\{\hat{\boldsymbol{g}}_k(\mathring{\boldsymbol{y}}_\Sigma)\}_{k=1}^m \cup \{\hat{\boldsymbol{n}}(\mathring{\boldsymbol{y}}_\Sigma)\}$ 作为 $\mathbb{R}^{m+1}$ 的单位正交基, $\{\hat{X}^k\}_{k=1}^{m+1}$ 为对应的 Cartesian 坐标, 则局部有

$$\begin{cases} \hat{X}^k = \Delta y_\Sigma^k + o^k(|\Delta\boldsymbol{y}_\Sigma|_{\mathbb{R}^m}), \quad k = 1, \cdots, m, \\ \hat{X}^{m+1} = \dfrac{1}{2}\left[\lambda_1(\Delta y_\Sigma^1)^2 + \cdots + \lambda_m(\Delta y_\Sigma^m)^2\right] \\ \qquad = \dfrac{1}{2}\left[\lambda_1(\hat{X}^1)^2 + \cdots + \lambda_m(\hat{X}^m)^2\right] + o(|\Delta\boldsymbol{y}_\Sigma|_{\mathbb{R}^m}^2) \\ \qquad = \dfrac{1}{2}\left[\lambda_1(\hat{X}^1)^2 + \cdots + \lambda_m(\hat{X}^m)^2\right] + o(|\Delta\hat{\boldsymbol{X}}_\Sigma|_{\mathbb{R}^m}^2). \end{cases}$$

亦即, 在二阶精度下, 曲面有局部 Monge 型表示

$$\mathbb{R}^m \ni \begin{pmatrix} \hat{X}^1 \\ \vdots \\ \hat{X}^m \end{pmatrix} \mapsto \begin{pmatrix} \hat{X}^1 \\ \vdots \\ \hat{X}^m \\ \hat{X}^{m+1} \end{pmatrix}(\hat{X}^1, \cdots, \hat{X}^m) = \begin{pmatrix} \hat{X}^1 \\ \vdots \\ \hat{X}^m \\ \dfrac{1}{2}\left[\lambda_1(\hat{X}^1)^2 + \cdots + \lambda_m(\hat{X}^m)^2\right] \end{pmatrix} \in \mathbb{R}^{m+1}.$$

受上述分析启发, 引入曲面 $\boldsymbol{\Sigma}(\boldsymbol{x}) \in \mathbb{R}^{m+1}$ 的另一参数 $\{y_\Sigma^i\}_{i=1}^m$, 满足

$$D\boldsymbol{y}_\Sigma(\boldsymbol{x}_\Sigma) = \boldsymbol{S}^{-1}(\boldsymbol{x}_\Sigma) \in \mathbb{R}^{m\times m},$$

式中 $\boldsymbol{S}(\boldsymbol{x}_\Sigma) \in \mathbb{R}^{m\times m}$ 非奇异, 成立

$$\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{x}_\Sigma)\left(g_{ij}\right)(\boldsymbol{x}_\Sigma)\boldsymbol{S}(\boldsymbol{x}_\Sigma) = \boldsymbol{I}_m, \quad \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{x}_\Sigma)\left(b_{ij}\right)(\boldsymbol{x}_\Sigma)\boldsymbol{S}(\boldsymbol{x}_\Sigma) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}.$$

亦即 $\boldsymbol{S}(\boldsymbol{x}_\Sigma)$ 为将 $\left(g_{ij}\right)(\boldsymbol{x}_\Sigma)$ 和 $\left(b_{ij}\right)(\boldsymbol{x}_\Sigma)$ 同时对角化的非奇异阵. 当 $\boldsymbol{S}(\boldsymbol{x}_\Sigma)$ 是足够光滑, 可有参数变换 $\boldsymbol{y}_\Sigma(\boldsymbol{x}_\Sigma)$ 为一定区域上的微分同胚. 曲面二组参数所确定的局部基, 如图1所示. 由

$$\hat{\boldsymbol{g}}_i(\boldsymbol{y}_\Sigma) \triangleq \frac{\partial\hat{\boldsymbol{\Sigma}}}{\partial y_\Sigma^i}(\boldsymbol{y}_\Sigma) = \frac{\partial x_\Sigma^k}{\partial y_\Sigma^i}\frac{\partial\boldsymbol{\Sigma}}{\partial x_\Sigma^k}(\boldsymbol{x}_\Sigma) = \frac{\partial x_\Sigma^k}{\partial y_\Sigma^i}\boldsymbol{g}_k(\boldsymbol{x}_\Sigma),$$

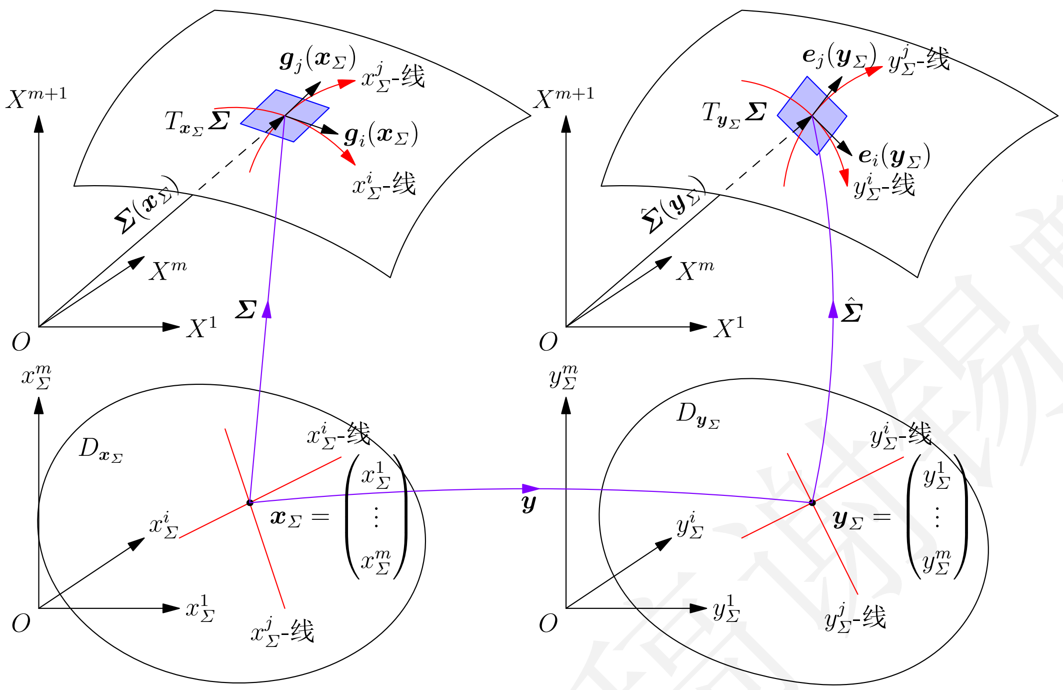

Figure 1: 曲面局部参数化示意

即有

$$\begin{pmatrix} \hat{\boldsymbol{g}}_1 & \cdots & \hat{\boldsymbol{g}}_m \end{pmatrix}(\boldsymbol{y}_\Sigma) = \begin{pmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m \end{pmatrix}(\boldsymbol{x}_\Sigma) D\boldsymbol{x}_\Sigma(\boldsymbol{y}_\Sigma) = \begin{pmatrix} \boldsymbol{g}_1 & \cdots & \boldsymbol{g}_m \end{pmatrix} \boldsymbol{S}(\boldsymbol{x}_\Sigma).$$

因此, 根据

$$\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{x}_\Sigma)\begin{pmatrix} g_{ij} \end{pmatrix}(\boldsymbol{x}_\Sigma)\boldsymbol{S}(\boldsymbol{x}_\Sigma) = \boldsymbol{I}_m$$

有 $\{\hat{\boldsymbol{g}}_i(\boldsymbol{y}_\Sigma) =: \boldsymbol{e}_i(\boldsymbol{y}_\Sigma)\}_{i=1}^m$ 为 $T_{\boldsymbol{y}}\boldsymbol{\Sigma}$ 的单位正交基. 亦即, $\hat{\boldsymbol{\Sigma}}(\boldsymbol{y}_\Sigma)$ 所诱导的切平面的局部协变基为单位正交基. 考虑

$$\begin{aligned} \hat{b}_{ij}(\boldsymbol{y}_\Sigma) &\triangleq \left(\frac{\partial \hat{\boldsymbol{g}}_j}{\partial y_\Sigma^i}(\boldsymbol{y}_\Sigma), \hat{\boldsymbol{n}}\right)_{\mathbb{R}^{m+1}} = -\left(\boldsymbol{g}_j(\boldsymbol{y}_\Sigma), \frac{\partial \hat{\boldsymbol{n}}}{\partial y_\Sigma^i}(\boldsymbol{y}_\Sigma)\right)_{\mathbb{R}^{m+1}} \\ &= -\left(\frac{\partial x_\Sigma^k}{\partial y_\Sigma^j}(\boldsymbol{y}_\Sigma)\boldsymbol{g}_k(\boldsymbol{x}_\Sigma), \frac{\partial x_\Sigma^i}{\partial y_\Sigma^i}(\boldsymbol{y}_\Sigma)\frac{\partial \boldsymbol{n}}{\partial x_\Sigma^l}(\boldsymbol{x}_\Sigma)\right)_{\mathbb{R}^{m+1}} \\ &= \frac{\partial x_\Sigma^k}{\partial y_\Sigma^j}(\boldsymbol{y}_\Sigma)\frac{\partial x_\Sigma^i}{\partial y_\Sigma^i}(\boldsymbol{y}_\Sigma)\left(\frac{\partial \boldsymbol{g}_k}{\partial x_\Sigma^l}(\boldsymbol{x}_\Sigma), \boldsymbol{n}\right)_{\mathbb{R}^{m+1}} \\ &= \frac{\partial x_\Sigma^k}{\partial y_\Sigma^j}(\boldsymbol{y}_\Sigma)\frac{\partial x_\Sigma^i}{\partial y_\Sigma^i}(\boldsymbol{y}_\Sigma) b_{lk}(\boldsymbol{x}_\Sigma), \end{aligned}$$

即有

$$\begin{aligned} \begin{pmatrix} \hat{b}_{ij} \end{pmatrix}(\boldsymbol{y}_\Sigma) &= (D\boldsymbol{x}_\Sigma)^{\mathrm{T}}(\boldsymbol{y}_\Sigma)\begin{pmatrix} b_{lk} \end{pmatrix}(\boldsymbol{x}_\Sigma) D\boldsymbol{x}_\Sigma(\boldsymbol{y}_\Sigma) \\ &= \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{y}_\Sigma)\begin{pmatrix} b_{lk} \end{pmatrix}(\boldsymbol{x}_\Sigma)\boldsymbol{S}(\boldsymbol{x}_\Sigma) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}. \end{aligned}$$

综上, 有

$$\hat{\boldsymbol{\Sigma}}(\boldsymbol{y}_\Sigma + \Delta \boldsymbol{y}_\Sigma) = \hat{\boldsymbol{\Sigma}}(\boldsymbol{y}_\Sigma) + \left[\Delta y_\Sigma^k + o^k(|\Delta \boldsymbol{y}_\Sigma|_{\mathbb{R}^{m+1}})\right] \hat{\boldsymbol{g}}_k(\boldsymbol{y}_\Sigma) + \frac{1}{2}\lambda_k(\boldsymbol{y}_\Sigma)(\Delta y_\Sigma^k)^2 \hat{\boldsymbol{n}}(\boldsymbol{y}_\Sigma) + \boldsymbol{o}(|\Delta \boldsymbol{y}_\Sigma|_{\mathbb{R}^{m+1}}).$$

现以 $\{\hat{\boldsymbol{g}}_i(\boldsymbol{y}_\Sigma) =: \boldsymbol{e}_i(\boldsymbol{y}_\Sigma)\}_{i=1}^m \cup \{\hat{\boldsymbol{n}}(\boldsymbol{y}_\Sigma)\}$ 作为 $\mathbb{R}^{m+1}$ 的单位正交基, $\{\hat{Y}^k\}_{k=1}^{m+1}$ 为对应的 Cartesian 坐标, 则局部有

$$\begin{cases} \hat{Y}^k = \Delta y_\Sigma^k + o^k(|\Delta \boldsymbol{y}_\Sigma|_{\mathbb{R}^{m+1}}), \quad k = 1, \cdots, m, \\ \hat{Y}^{m+1} = \dfrac{1}{2}\left[\lambda_1(\boldsymbol{y}_\Sigma)(\hat{Y}^1)^2 + \cdots + \lambda_m(\boldsymbol{y}_\Sigma)(\hat{Y}^m)^2\right] + o(|\Delta \boldsymbol{y}_\Sigma|^2_{\mathbb{R}^{m+1}}). \end{cases}$$

2 应用事例

3 建立路径

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